The Complex Potential Approach to Power-Logarithmic Stress Singularities for V-Notched Cracks in a Bi-Material

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Power-logarithmic stress singularites and the coefficient vectors for V-notched cracks in a bi-material are obtained by using complex potentials and the concept of repeated roots for general solutions. On several examples, it is shown that the results obtained using the complex potential approach are identical to those found by Bogy (1970) using the Mellin transform method, and to those found by Dempsey and Sinclair (1979, 1981) using the Airy stress function approach.

Key Words: Bi-Material, V-Notched Crack, Power-Logarithmic Stress Singularity, Complex Potential.

1. Introduction

Stress singularities for V-notched cracks in a bi-material are of interest for many engineers. Bogy (1970, 1971) commenced research on the stress singularity problem of a wedge in dissimilar materials. His work was followed by others: Dunders and Lee (1972), Hein and Erdogan (1971), Carpenter and Byers (1987) etc.

These works focused mainly on the power stress singularity. Logarithmic stress singularities were briefly discussed in Bogy (1970, 1971). Power-logarithmic stress singularities were reported by Dempsey and Sinclair (1979, 1981) where it was indicated that the logarithmic singularity is only a special case of power-logarithmic singularities. Dempsey (1995) reported specific cases which have power-logarithmic singularities even for homogeneous boundary conditions.

There are several available methods for singularity analysis, each with as own merits and demerits. The Mellin transform method is an elegant but complicated method of examining stress singularities. Hein and Erdogan (1971) used the Mellin transform method to examine power singularities and Bogy (1970, 1971) used the Mellin transform method to examine power and power-logarithmic stress singularities. The straightforward Airy stress function method was used by Williams (1952) to examine power stress singularities. The straightforward complex potential method has been used by England (1971), Stern and Soni (1976), and Carpenter and Byers (1987) to examine power stress singularities. To the authors knowledge, it has not yet been applied to analyze power-logarithmic stress singularities. In this paper, power-logarithmic singularities for V-notched cracks in a bi-material are investigated using the complex potential method and the concept of repeated roots for the general solution.

2. Basic Equations and the Power Stress Singularity for V-Notched Cracks in a Bi-Material

In the case of plane isotropic elasticity in the absence of body forces as shown in Fig. 1, the stresses and displacements can be expressed in terms of complex potentials $\phi_j(z)$ and $\psi_j(z)$ as follows (Carpenter and Byers, 1987),

$$u_{jr}+iu_{j\theta}=(2\mu_j)^{-1}e^{-i\theta}[\chi_j\phi_j(z)-z\bar{\phi}_j'(\bar{z})-\bar{\psi}_j(\bar{z})], \quad (1)$$

$$\sigma_{jrr}+i\sigma_{jr\theta}=\phi_j'(z)+\bar{\phi}_j'(\bar{z})-\bar{z}\bar{\phi}_j''(\bar{z})-\bar{z}z^{-1}\bar{\psi}'(\bar{z}), \quad (2)$$

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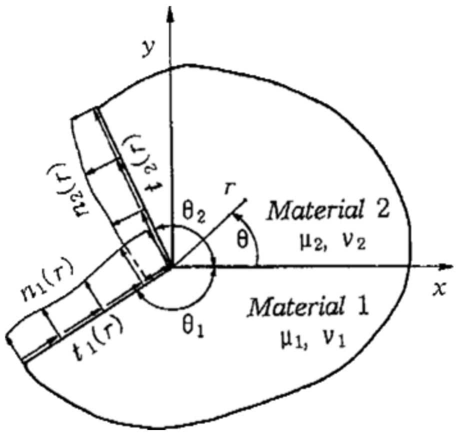

Fig. 1 V-notched crack in a bi-material.

$$\sigma_{j\theta\theta}-i\sigma_{jr\theta}=\phi_j'(z)+\bar{\phi}_j'(\bar{z})+\bar{z}\bar{\phi}_j''(\bar{z})+\bar{z}z^{-1}\bar{\psi}'(\bar{z}), \tag{3}$$

where $z=re^{i(\theta)}$, μ_j is the shear modulus, ν_j is Poisson's ratio,

$$\begin{cases}\chi_j=3-4\nu_j \quad (\textit{plane strain}),\\ \chi_j=\dfrac{3-\nu_j}{1+\nu_j} \quad (\textit{plane stress})\end{cases} \tag{4}$$

and a bar over a symbol $(-)$ denotes the complex conjugate of the symbol. The subscript j refers to material j.

Assuming the following complex potentials, the stresses and displacements for a V-notched crack in a bi-material were obtained as (Carpenter and Byers, 1987)

$$\phi_j(z)=A_jz^{\lambda}+a_jz^{\bar{\lambda}}, \tag{5}$$

$$\psi_j(z)=B_jz^{\lambda}+b_jz^{\bar{\lambda}} \tag{6}$$

where λ, A_j, a_j, B_j and b_j are assumed to be complex.

Substituting Eqs. (5) and (6) into Eqs. (1), (2) and (3), the power stresses and displacements are obtained (Carpenter and Byers, 1987)

$$\begin{aligned}u_{jr}+iu_{j\theta}=(2\mu_j)^{-1}[&r^{\lambda}\{\chi_jA_je^{i\theta(\lambda-1)}-\bar{a}_j\lambda e^{i\theta(-\lambda+1)}-\bar{b}_je^{i\theta(-\lambda-1)}\}\\&+r^{\bar{\lambda}}\{\chi_ja_je^{i\theta(\bar{\lambda}-1)}-\bar{A}_j\bar{\lambda}e^{i\theta(-\bar{\lambda}+1)}-\bar{B}_je^{i\theta(-\bar{\lambda}-1)}\}],\end{aligned} \tag{7}$$

$$\begin{aligned}\sigma_{jrr}+i\sigma_{jr\theta}=&r^{\lambda-1}\{A_j\lambda e^{i\theta(\lambda-1)}-\bar{a}_j\lambda^2e^{i\theta(-\lambda+1)}-\bar{b}_j\lambda e^{i\theta(-\lambda-1)}+2\bar{a}_j\lambda e^{i\theta(-\lambda+1)}\}\\&+r^{\bar{\lambda}-1}\{a_j\bar{\lambda}e^{i\theta(\bar{\lambda}-1)}-\bar{A}_j\bar{\lambda}^2e^{i\theta(-\bar{\lambda}+1)}-\bar{B}_j\bar{\lambda}e^{i\theta(-\bar{\lambda}-1)}+2\bar{A}_j\bar{\lambda}e^{i\theta(-\bar{\lambda}+1)}\},\end{aligned} \tag{8}$$

$$\begin{aligned}\sigma_{j\theta\theta}-i\sigma_{jr\theta}=&r^{\lambda-1}\{A_j\lambda e^{i\theta(\lambda-1)}+\bar{a}_j\lambda^2e^{i\theta(-\lambda+1)}+\bar{b}_j\lambda e^{i\theta(-\lambda-1)}\}+r^{\bar{\lambda}-1}\{a_j\bar{\lambda}e^{i\theta(\bar{\lambda}-1)}\\&+\bar{A}_j\bar{\lambda}^2e^{i\theta(-\bar{\lambda}+1)}+\bar{B}_j\bar{\lambda}e^{i\theta(-\bar{\lambda}-1)}\}.\end{aligned} \tag{9}$$

The boundary conditions for Fig. 1 are as follows:

$$\sigma_{1\theta\theta}-i\sigma_{1r\theta}=n_1(r)+it_1(r) \quad \text{at } \theta=-\theta_1, \tag{10}$$

$$\sigma_{1\theta\theta}-i\sigma_{1r\theta}=\sigma_{2\theta\theta}-i\sigma_{2r\theta} \quad \text{at } \theta=0, \tag{11}$$

$$u_{1r}+iu_{1\theta}=u_{2r}+iu_{2\theta} \quad \text{at } \theta=0, \tag{12}$$

$$\sigma_{2\theta\theta}-i\sigma_{2r\theta}=n_2(r)+it_2(r) \quad \text{at } \theta=\theta_2. \tag{13}$$

Entering Eqs. (7), (8) and (9) into Eqs. (10), (11), (12) and (13), the following equation can be obtained after some manipulation if only the homogeneous equations are considered:

$$[B]\{A\}=\{0\} \tag{14}$$

where

$$\{A\}=[A_1\ \bar{a}_1\ B_1\ \bar{b}_1\ A_2\ \bar{a}_2\ B_2\ \bar{b}_2]^T, \tag{15}$$

$$\{0\}=[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T \tag{16}$$

in which the superscript T stands for the transpose and

$$[B]=\begin{bmatrix}s_{11} & 0\\ s_{12} & s_{21}\\ u_{12} & u_{21}\\ 0 & s_{22}\end{bmatrix} \tag{17}$$

where

$$[s_{11}]=\begin{bmatrix}\lambda^2e^{-i(-\theta_1)(-\lambda+1)} & \lambda e^{-i(-\theta_1)(\lambda-1)} & \lambda e^{-i(-\theta_1)(-\lambda-1)} & 0\\ \lambda e^{i(-\theta_1)(\lambda-1)} & \lambda^2e^{i(-\theta_1)(-\lambda+1)} & 0 & \lambda e^{i(-\theta_1)(-\lambda-1)}\end{bmatrix},$$

$$[s_{12}]=\begin{bmatrix}\lambda^2 & \lambda & \lambda & 0\\ \lambda & \lambda^2 & 0 & \lambda\end{bmatrix},$$

$$[s_{21}]=\begin{bmatrix}-\lambda^2 & -\lambda & -\lambda & 0\\ -\lambda & -\lambda^2 & 0 & -\lambda\end{bmatrix},$$

$$[u_{12}]=\begin{bmatrix}-\dfrac{\lambda}{2\mu_1} & \dfrac{\chi_1}{2\mu_1} & -\dfrac{1}{2\mu_1} & 0\\ \dfrac{\chi_1}{2\mu_1} & -\dfrac{\lambda}{2\mu_1} & 0 & -\dfrac{1}{2\mu_1}\end{bmatrix},$$

$$[u_{21}]=\begin{bmatrix}\dfrac{\lambda}{2\mu_2} & -\dfrac{\chi_2}{2\mu_2} & \dfrac{1}{2\mu_2} & 0\\ -\dfrac{\chi_2}{2\mu_2} & \dfrac{\lambda}{2\mu_2} & 0 & \dfrac{1}{2\mu_2}\end{bmatrix},$$

$$[s_{22}]=\begin{bmatrix}\lambda^2e^{-i\theta_2(-\lambda+1)} & \lambda e^{-i\theta_2(\lambda-1)} & \lambda e^{-i\theta_2(-\lambda-1)} & 0\\ \lambda e^{i\theta_2(\lambda-1)} & \lambda^2e^{i\theta_2(-\lambda+1)} & 0 & \lambda e^{i\theta_2(-\lambda-1)}\end{bmatrix}.$$

For nontrivial solutions for Eq. (14),

$$|B|=D(\lambda)=0 \tag{18}$$

where $D(\lambda)$ denotes the determinant of matrix [B]. The characteristic Eq. (18) for the homogeneous equations can have single and repeated roots.

For the inhomogeneous equations, the particular solutions must be found; that is, λ for the particular solutions must be obtained.

The general solution then includes the solution of the homogeneous equations and the particular solution. If the repeated roots exist in the general solution, the complex potentials of Eqs. (5) and (6) must be reconsidered.

3. The Power-Logarithmic Singularity by the Complex Potential Approach for Repeated Roots

If λ for the general solution is a repeated root, the complex potentials of Eqs. (5) and (6) must be reconsidered. Dempsey (1995) has shown that power-logarithmic stress singularities occur on transition loci separating real and complex zeros for the eigenvalue. Thus, power-logarithmic singularities occur when λ is real. Therefore, it can be easily obtained by differentiating Eqs. (5) and (6) with respect to λ that the additional complex potentials have the following forms for a double root:

$$\hat{\phi}_j(z)=\hat{A}_j z^{\lambda}\ln z, \tag{19}$$
$$\hat{\psi}_j(z)=\hat{B}_j z^{\lambda}\ln z. \tag{20}$$

where

$$\hat{A}_j=\hat{A}_{j1}+i\hat{A}_{j2},$$
$$\hat{B}_j=\hat{B}_{j1}+i\hat{B}_{j2}.$$

By differentiating Eqs. (19) and (20) with respect to λ, the additional complex potentials for a triple root can be also obtained as follows:

$$\hat{\hat{\phi}}_j(z)=\hat{\hat{A}}_j z^{\lambda}(\ln z)^2, \tag{21}$$
$$\hat{\hat{\psi}}_j(z)=\hat{\hat{B}}_j z^{\lambda}(\ln z)^2. \tag{22}$$

Substituting Eqs. (19) and (20) into Eqs. (1), (2) and (3) gives the stresses and displacements of the power-logarithmic singularity for a double root.

Substituting Eqs. (19) and (20) into Eq. (1) gives the following additional displacements for a double root:

$$\begin{aligned}\hat{u}_{jr}=-\frac{r^{\lambda}}{2\mu_j}\Big[&\ln r[(\lambda-\chi_j)\{\hat{A}_{j1}\cos(\lambda-1)\theta\\&-\hat{A}_{j2}\sin(\lambda-1)\theta\}+\hat{B}_{j1}\cos(\lambda+1)\theta\\&-\hat{B}_{j2}\sin(\lambda+1)\theta]+\theta[(\lambda-\chi_j)\\&\{\hat{A}_{j1}\sin(\lambda-1)\theta-\hat{A}_{j2}\cos(\lambda-1)\theta\}\\&-\hat{B}_{j1}\sin(\lambda+1)\theta-\hat{B}_{j2}\cos(\lambda+1)\theta]\\&+\hat{A}_{j1}\cos(\lambda-1)\theta-\hat{A}_{j2}\sin(\lambda-1)\theta\Big],\end{aligned} \tag{23}$$

$$\begin{aligned}\hat{u}_{j\theta}=-\frac{r^{\lambda}}{2\mu_j}\Big[&\ln r[(\lambda-\chi_j)\{\hat{A}_{j1}\sin(\lambda-1)\theta\\&-\hat{A}_{j2}\cos(\lambda-1)\theta\}+\hat{B}_{j1}\sin(\lambda+1)\\&+\hat{B}_{j2}\cos(\lambda-1)\theta]+\theta[(\lambda+\chi_j)\\&\{\hat{A}_{j1}\cos(\lambda-1)\theta-\hat{A}_{j2}\sin(\lambda-1)\theta\}\\&+\hat{B}_{j1}\cos(\lambda+1)\theta-\hat{B}_{j2}\sin(\lambda+1)\theta]\\&+\hat{A}_{j1}\sin(\lambda-1)\theta-\hat{A}_{j2}\cos(\lambda-1)\theta\Big],\end{aligned} \tag{24}$$

By substituting Eqs. (19) and (20) into Eqs. (2) and (3), the additional stresses for a double root can be obtained as follows:

$$\begin{aligned}\hat{\sigma}_{jrr}=-r^{\lambda-1}\Big[&\lambda\ln r[(\lambda-3)\{\hat{A}_{j1}\cos(\lambda-1)\theta\\&-\hat{A}_{j2}\sin(\lambda-1)\theta\}+\hat{B}_{j1}\cos(\lambda+1)\theta\\&-\hat{B}_{j2}\sin(\lambda+1)\theta]+\lambda\theta[(\lambda-3)\\&\{-\hat{A}_{j1}\sin(\lambda-1)\theta-\hat{A}_{j2}\cos(\lambda-1)\theta\},\\&-\hat{B}_{j1}\sin(\lambda+1)\theta-\hat{B}_{j2}\cos(\lambda+1)\theta]\\&+(2\lambda-3)\{\hat{A}_{j1}\cos(\lambda-1)\theta\\&-\hat{A}_{j2}\sin(\lambda-1)\theta\}+\hat{B}_{j1}\cos(\lambda+1)\theta\\&-\hat{B}_{j2}\sin(\lambda+1)\theta\Big],\end{aligned} \tag{25}$$

$$\begin{aligned}\hat{\sigma}_{j\theta\theta}=r^{\lambda-1}\Big[&\lambda\ln r[(\lambda+1)\{\hat{A}_{j1}\cos(\lambda-1)\theta\\&-\hat{A}_{j2}\sin(\lambda-1)\theta\}+\hat{B}_{j1}\cos(\lambda+1)\theta\\&-\hat{B}_{j2}\sin(\lambda+1)\theta]+\lambda\theta[(\lambda+1)\\&\{-\hat{A}_{j1}\sin(\lambda-1)\theta-\hat{A}_{j2}\cos(\lambda-1)\theta\}\\&-\hat{B}_{j1}\sin(\lambda+1)\theta-\hat{B}_{j2}\cos(\lambda+1)\theta]\\&+(2\lambda+1)\{\hat{A}_{j1}\cos(\lambda-1)\theta\\&-\hat{A}_{j2}\sin(\lambda-1)\theta\}+\hat{B}_{j1}\cos(\lambda+1)\theta\\&-\hat{B}_{j2}\sin(\lambda+1)\theta\Big],\end{aligned} \tag{26}$$

$$\begin{aligned}\hat{\sigma}_{jr\theta}=r^{\lambda-1}\Big[&\lambda\ln r[(\lambda-1)\{\hat{A}_{j1}\sin(\lambda-1)\theta\\&-\hat{A}_{j2}\cos(\lambda-1)\theta\}+\hat{B}_{j1}\sin(\lambda+1)\theta\\&-\hat{B}_{j2}\cos(\lambda+1)\theta]+\lambda\theta[(\lambda-1)\\&\{\hat{A}_{j1}\cos(\lambda-1)\theta-\hat{A}_{j2}\sin(\lambda-1)\theta\}\\&+\hat{B}_{j1}\sin(\lambda+1)\theta-\hat{B}_{j2}\cos(\lambda+1)\theta]\\&+(2\lambda-1)\{\hat{A}_{j1}\sin(\lambda-1)\theta\\&+\hat{A}_{j2}\cos(\lambda-1)\theta\}+\hat{B}_{j1}\sin(\lambda+1)\theta\\&-\hat{B}_{j2}\cos(\lambda+1)\theta\Big].\end{aligned} \tag{27}$$

These results are equal to the results of Sinclair (1996). For a triple root, the additional displacements and stresses (which are also the same as those obtained in Sinclair (1996)) can be also obtained by differentiating the displacement Eqs. (23), (24) and stresses Eqs. (25), (26) and (27).

4. Coefficient Vectors for a Repeated Root

4.1 Coefficient vectors for a double root

Consider the homogeneous boundary condition problem. If the characteristic Eq. (18) has a double root, the complex potentials of Eq. (19) should be added to Eq. (5), and Eq. (20) should be added to Eq. (6). The stress and displacement fields for these complex potentials have power-logarithmic singularity. Entering these stresses and displacements into the boundary conditions of Eqs. (10), (11), (12), (13) and manipulating gives the following equation for the homogeneous boundary conditions:

$$r^{\lambda-1}\ln r[B]\{\hat{A}\}+r^{\lambda-1}[G]\{\hat{A}\} + r^{\lambda-1}[B]\{A\}=\{0\} \tag{28}$$

where

$$\{\hat{A}\}=[\hat{A}_1\ \hat{\bar{a}}_1\ \hat{B}_1\ \hat{\bar{b}}_1\ \hat{A}_2\ \hat{\bar{a}}_2\ \hat{B}_2\ \hat{\bar{b}}_2]^T \tag{29}$$

matrix [B] is the same of Eq. (17) and

$$[G]=\frac{d[B]}{d\lambda}. \tag{30}$$

Therefore, the following equations must be satisfied:

$$[B]\{\hat{A}\}=0, \tag{31}$$

$$[G]\{\hat{A}\}+[B]\{A\}=0. \tag{32}$$

Secondly, consider the inhomogeneous boundary condition problem. If the characteristic Eq. (18) has a single root and the particular solution has the same root, this case can be treated as a double root case. The following equation can be obtained:

$$r^{\lambda-1}\ln r[B]\{\hat{A}\}+r^{\lambda-1}[G]\{\hat{A}\} + r^{\lambda-1}[B]\{A\}=\{f\}. \tag{33}$$

where

$$\{f\}=[n_1(r)-it_1(r)\ \ n_1(r)+it_1(r)\ \ 0\ 0\ 0\ 0\ n_2(r)-it_2(r)\ \ n_2(r)+it_2(r)]^T. \tag{34}$$

4.2 Coefficient Vectors for a Triple Root

Consider the homogeneous boundary condition problem. If the characteristic Eq. (18) has a triple root, the complex potentials of Eqs. (19) and (21) should be added to Eq. (5), and Eqs. (20) and (22) should be added to Eq. (6). Entering the stresses and displacements for these complex potentials into the boundary conditions of Eqs. (10), (11), (12), (13) and manipulating gives the following equation for the homogeneous boundary conditions:

$$r^{\lambda-1}(\ln r)^2[B]\{\hat{\hat{A}}\}+r^{\lambda-1}\ln r[G]\{\hat{\hat{A}}\} + r^{\lambda-1}\ln r[B]\{\hat{A}\}+r^{\lambda-1}[H]\{\hat{\hat{A}}\} + r^{\lambda-1}[G]\{\hat{A}\}+r^{\lambda-1}[B]\{A\}=\{0\} \tag{35}$$

where

$$\{\hat{\hat{A}}\}=[\hat{\hat{A}}_1\ \hat{\hat{\bar{a}}}_1\ \hat{\hat{B}}_1\ \hat{\hat{\bar{b}}}_1\ \hat{\hat{A}}_2\ \hat{\hat{\bar{a}}}_2\ \hat{\hat{B}}_2\ \hat{\hat{\bar{b}}}_2]^T \tag{36}$$

and

$$[H]=\frac{d[G]}{d\lambda}. \tag{37}$$

Therefore, the following equations must be satisfied:

$$[B]\{\hat{\hat{A}}\}=0, \tag{38}$$

$$[G]\{\hat{\hat{A}}\}+[B]\{\hat{A}\}=0, \tag{39}$$

$$[H]\{\hat{\hat{A}}\}+[G]\{\hat{A}\}+[B]\{A\}=0. \tag{40}$$

Now, let us consider the inhomogeneous boundary condition problem. If the characteristic Eq. (18) has a double root and the particular solution has the same root, this case can be treated as a triple root case. The following equation can be obtained:

$$r^{\lambda-1}(\ln r)^2[B]\{\hat{\hat{A}}\}+r^{\lambda-1}\ln r[G]\{\hat{\hat{A}}\} + r^{\lambda-1}[B]\{\hat{A}\}+r^{\lambda-1}[H]\{\hat{\hat{A}}\} + r^{\lambda-1}[G]\{\hat{A}\}+r^{\lambda-1}[B]\{A\}=\{f\} \tag{41}$$

where

$$\{f\}=[n_1(r)-it_1(r)\ \ n_1(r)+it_1(r)\ \ 0\ 0\ 0\ 0\ n_2(r)-it_2(r)\ \ n_2(r)+it_2(r)]^T. \tag{42}$$

5. The Coefficient Vector for the Real Root λ

When the root λ is real, the coefficient vectors $\{A\}$, $\{\hat{A}\}$ and $\{\hat{\hat{A}}\}$ of Eqs. (14), (28), (33) and (35) become as follows:

$$\{A\}=[A_1\ \bar{A}_1\ B_1\ \bar{B}_1\ A_2\ \bar{A}_2\ B_2\ \bar{B}_2]^T, \quad (43)$$
$$\{\hat{A}\}=[\hat{A}_1\ \bar{\hat{A}}_1\ \hat{B}_1\ \bar{\hat{B}}_1\ \hat{A}_2\ \bar{\hat{A}}_2\ \hat{B}_2\ \bar{\hat{B}}_2]^T, \quad (44)$$
$$\{\hat{\hat{A}}\}=[\hat{\hat{A}}_1\ \bar{\hat{\hat{A}}}_1\ \hat{\hat{B}}_1\ \bar{\hat{\hat{B}}}_1\ \hat{\hat{A}}_2\ \bar{\hat{\hat{A}}}_2\ \hat{\hat{B}}_2\ \bar{\hat{\hat{B}}}_2]^T, \quad (45)$$

where

$$A_j=A_{j1}+iA_{j2},\ B_j=B_{j1}+iB_{j2},$$
$$\hat{\hat{A}}_i=\hat{\hat{A}}_{j1}+i\hat{\hat{A}}_{i2},\ \hat{\hat{B}}_j=\hat{\hat{B}}_{j1}+i\hat{\hat{B}}_{j2}.$$

Then, Eqs. (43), (44) and (45) become as follows:

$$\{A\}=[I]\{A_r\}, \quad (46)$$
$$\{\hat{A}\}=[I]\{\hat{A}_r\}, \quad (47)$$
$$\{\hat{\hat{A}}\}=[I]\{\hat{\hat{A}}_r\} \quad (48)$$

where

$$\{A_r\}=[A_{11}\ A_{12}\ B_{11}\ B_{12}\ A_{21}\ A_{22}\ B_{21}\ B_{22}]^T, \quad (49)$$
$$\{\hat{A}_r\}=[\hat{A}_{11}\ \hat{A}_{12}\ \hat{B}_{11}\ \hat{B}_{12}\ \hat{A}_{21}\ \hat{A}_{22}\ \hat{B}_{21}\ \hat{B}_{22}]^T, \quad (50)$$
$$\{\hat{\hat{A}}_r\}=[\hat{\hat{A}}_{11}\ \hat{\hat{A}}_{12}\ \hat{\hat{B}}_{11}\ \hat{\hat{B}}_{12}\ \hat{\hat{A}}_{21}\ \hat{\hat{A}}_{22}\ \hat{\hat{B}}_{21}\ \hat{\hat{B}}_{22}]^T, \quad (51)$$

$$[I]=\begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i \end{bmatrix}. \quad (52)$$

For an example, substituting Eqs. (46), (47) and (48) into Eqs. (38), (39) and (40) gives the following equations:

$$[B]\{\hat{\hat{A}}\}=[B_I]\{\hat{\hat{A}}_r\}=0, \quad (53)$$
$$[G]\{\hat{\hat{A}}\}+[B]\{\hat{A}\}=[G_I]\{\hat{\hat{A}}_r\} + [B_I]\{\hat{A}_r\}=0, \quad (54)$$
$$[H]\{\hat{\hat{A}}\}+[G]\{\hat{A}\}+[B]\{A\}=[H_I]\{\hat{\hat{A}}_r\} + [G_I]\{\hat{A}_r\}+[B_I]\{A_r\}=0, \quad (55)$$

where

$$[B_I]=[B][I],$$
$$[G_I]=[G][I],$$
$$[H_I]=[H][I].$$

6. Examples and Discussion

Cases having the power-logarithmic stress singularity for homogeneous boundary conditions as reported by Dempsey (1995) are considered.

In Fig. 1, when the conditions are $\theta_1=\theta_2=160°$, $\nu_1=\nu_2=0.2$, $E_2/E_1=2.72525$, and plane strain and the boundary conditions are traction free, the characteristic Eq. (18) has double root given by $\lambda=\lambda_o=0.56983$ (because $D(\lambda_o)=0$ and $(\frac{dD}{d\lambda})_{\lambda=\lambda_o}=0.$). In another case, when $\theta_1=180°$, $\theta_2=90°$, $\nu_1=\nu_2=0.2$, $E_2/E_1=9.33084$, and plane strain and the boundary conditions are traction free, the characteristic Eq. (18) also has double root given by $\lambda=\lambda_o=0.56435$ (because $D(\lambda_o)=0$ and $(\frac{dD}{d\lambda})_{\lambda=\lambda_o}=0.$).

The previous two cases are examples having a power-logarithmic stress singularity for homogeneous boundary conditions and real roots. Therefore, from Eqs. (31) and (32), the following equations can be obtained:

$$[B_I]\{\hat{A}_r\}=0, \quad (56)$$
$$[G_I]\{\hat{A}_r\}+[B_I]\{A_r\}=0. \quad (57)$$

Because the two cases are $|G_I|\neq 0$, the following equation can be obtained from Eq. (57):

$$\{\hat{A}_r\}=-[G_I]^{-1}[B_I]\{A_r\}. \quad (58)$$

Entering Eq. (58) into Eq. (56) gives the following equation:

$$[Q_I]\{A_r\}=0 \quad (59)$$

where

$$[Q_I]=-[B_I][G_I]^{-1}[B_I]. \quad (60)$$

For a nontrivial solution for Eq. (59),

$$|Q_I|=0. \quad (61)$$

After obtaining $\{A_r\}$ from Eq. (59), $\{\hat{A}_r\}$ can be obtained from Eq. (58). Therefore, these cases are examples which have a power-logarithmic stress singularity for homogeneous boundary conditions. But if the matrices $[B_I]$ and $[Q_I]$ are similar matrices, the coefficient vector $\{\hat{A}_r\}$ will be zero and power-logarithmic stress singularities for homogeneous boundary conditions will not occur.

For inhomogeneous boundary conditions, an example having a logarithmic stress singularity as examined is Bogy's paper (1970) is considered. If the boundary conditions are given by constant tractions in Fig. 1 and the other conditions are θ_1

$=\theta_2=90°$, $\nu_1=0.3$, $\nu_2=0.1$, $E_1=229.38$, $E_2=64.73$ and plane strain, then, Dunders' parameters will be $\alpha=0.58821$ and $\beta=0.29411$, and the condition of $\alpha(\alpha-2\beta)=0$ will be satisfied. Therefore, the characteristic Eq. (18) does not have roots in the range $0<Re(\lambda)<1$. But, the characteristic Eq. (18) has a double root at $\lambda=\lambda_o=1$ because $D(\lambda_o)=0$ and $(\frac{dD}{d\lambda})_{\lambda=\lambda_o}=0$ are satisfied.

If the boundary conditions are given constant tractions as follows;

$$n_1(a)+it_1(a)=n_1+it_1, \tag{62}$$
$$n_2(a)+it_2(a)=n_2+it_2 \tag{63}$$

then λ of the particular solution is $\lambda=\lambda_o=1$. This problem is a case of a triple root for the general solution and the root λ is real. In this case the following equations are obtained from Eq. (41):

$$[B_I]\{\hat{\hat{A}}_r\}=0, \tag{64}$$
$$[G_I]\{\hat{\hat{A}}_r\}+[B_I]\{\hat{A}_r\}=0, \tag{65}$$
$$[H_I]\{\hat{\hat{A}}_r\}+[G_I]\{\hat{A}_r\}+[B_I]\{A_r\}=\{f\} \tag{66}$$

where

$$\{f\}=[n_1-it_1\ \ n_1+it_1\ \ 0\,0\,0\,0\ \ n_2-it_2\ \ n_2+it_2]^T. \tag{67}$$

Because this case corresponds to $|G_I|\neq 0$, the following equation can be obtained from Eq. (66):

$$\{\hat{A}_r\}=-[G_I]^{-1}[H_I]\{\hat{\hat{A}}_r\}-[G_I]^{-1}[B_I]\{A_r\}+[G_I]^{-1}\{f\}. \tag{68}$$

Entering Eq. (68) into Eq. (65) gives the following equation:

$$\{\hat{\hat{A}}_r\}=[P_I]^{-1}[Q_I]\{A_r\}-[P_I]^{-1}[B_I][G_I]^{-1}\{f\}. \tag{69}$$

where

$$[P_I]=[G_I]-[B_I][G_I]^{-1}[H_I]. \tag{70}$$

Entering Eq. (69) into Eq. (64) gives the following equation:

$$[R_I]\{A_r\}=[S_I]\{f\}. \tag{71}$$

where

$$[R_I]=[B_I][P_I]^{-1}[Q_I], \tag{72}$$
$$[S_I]=[B_I][P_I]^{-1}[B_I][G_I]^{-1}. \tag{73}$$

From Eqs. (71), (69) and (68), the coefficient vectors $\{A_r\}$, $\{\hat{\hat{A}}_r\}$ and $\{\hat{A}_r\}$ can be obtained. If the matrices $[B_I]$ and $[R_I]$ are similar matrices, the coefficient vectors $\{\hat{A}_r\}$ and $\{\hat{\hat{A}}_r\}$ will depend on the traction vector {f}. By this procedure, the results of Eqs. (4. 6) and (4. 7) of Bogy's paper (1970) may be obtained.

7. Conclusion

Power-logarithmic stress singularities are examined and the coefficient vectors for V-notched cracks in a bi-material are developed using the method of complex potentials and the concept of repeated roots for general solutions. Results obtained the several examples were identical to those found by Bogy (1970) using the Mellin transform method and to those found by Dempsey and Sinclair (1979, 1981) using the Airy stress function approach.

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